A flexible short-rate based four factor arbitrage-free term structure model with an explicit monetary policy rule

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Abstract

I derive an arbitrage-free four-factor term structure model that facilitates direct parametrization of the short-term interest rate process. The interplay between macroeconomic variables and the term structure via a monetary policy reaction function, in the spirit of Taylor (1993), is therefore directly supported. I show that the proposed model is a constrained member of the canonical GDTSM family proposed by Joslin, Singleton, and Zhu (2011). The model's loading structure bears close resemblance to that of the Svensson and Söderlind (1997) model, but it relies only on a single non-linear shape parameter, and the model is therefore easy to estimate. An empirical application to US data covering the period from 1961 to 2017 demonstrates that the proposed model fits yields well, and that an embedded policy rule, including industrial production and the inflation rate, is statistically significant and economically meaningful during this time-period.

Keywords: Yield Curve Modelling, Dynamic Svensson-Söderlind Model, central bank reaction function.

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1 Introduction

Financial market practitioners and central bankers rely on the yield curve and it's dynamic evolution to provide information about investor's expectations to future interest rates and to the uncertainty surrounding these expectations. Typically some version of a multi-factor yield curve model is used to extract such market based gauges, for example, the Nelson and Siegel (1987) and Svensson and Söderlind (1997) models are popular among market practitioners, while arbitrage-free models such as, Christensen, Diebold, and Rudebusch (2011), Adrian, Crump, and Mönch (2013), and Joslin, Singleton, and Zhu (2011), appeal to academics, given their internal model consistency - central banks rely on both type of models, probably given their public mandate and thus a desire to avoid reliance on one particular modeling approach.

Since yields observed at different maturities tend to co-move a parsimonious description of a panel of yields spanning time and maturity, can be obtained by modeling only a few underlying factors, see e.g. Litterman and Scheinkman (1991). Depending on how the factors are identified, they may have a particular interpretation as level, slope and curvature (e.g. Diebold and Li (2006)), or they may be pure principal components extracted from the data (e.g. Joslin, Singleton, and Zhu (2011)). However, regardless of the method used to extract the yield curve factors, they typically correlate very highly with empirical counterparts of the level, slope and curvature of the yield curve.[1]

For typical central bank and asset managers purposes the actual factor identification scheme is of little relevance, since the most important thing is to model yields with small approximation errors, and to have a good model for the dynamic evolution of the yield curve factors that produce good forecasts. However, this is not the case when the objective

[1]For a non-exhaustive, but representative sample, see Litterman and Scheinkman (1991), Joslin, Singleton, and Zhu (2011), Adrian, Crump, and Mönch (2013), Hamilton and Wu (2012) Diebold and Rudebusch (2013), Krippner (2015), Nelson and Siegel (1987), Svensson and Söderlind (1997), Diebold and Li (2006). A few exceptions exist: Nyholm (2017) shows how the traditional Nelson and Siegel (1987) factor structure can be rotated to represent short rate, slope and curvature (as opposed to level, (negative) slope, curvature); Creal and Wu (2017) derive a model where the factor structure comprises the short rate, the average expected future short rate (at a pre-selected maturity), and the term premium (at a pre-selected maturity); and Nyholm (2015) suggests an arbitrage-free three-factor model, with a factor interpretation identical to Nyholm (2017). Empirical counterparts of the yield curve factors denoted by level, slope, and curvature are typically the observed 10 year yield, the difference between the 3 month and 10 year yields, and two times the 3 year rate minus the 10 year and 3 month yields, respectively.

is to embed a monetary policy reaction function into the model, where macroeconomic variables directly impact the policy rate, and thereby the short end of the yield curve. In this case it is paramount to be able to specify how the short end of the yield curve correlates with lagged macroeconomic gauges for inflation and economic growth. And, it is therefore necessarily to impose some form of economic identification on the yield curve factors. This point is forcefully made by Joslin, Le, and Singleton (2013)[section 3]. They show that the workhorse latent-factor term-structure model leaves the short rate equation unidentified, in an economic sense, since the extracted yield curve factor(s) themselves are without economic meaning. When combining one or more latent factors with macro economic variables in the model's short rate equation, it is unclear how to attach economic meaning to this relationship, and thus, how to draw policy conclusions from it.

The purpose of the current paper is to address this issue in the context of an arbitrage-free Nelson-Siegel type setting. In doing so, the paper contributes to the literature in the following ways: (a) it develops an arbitrage-free discrete-time four-factor yield curve model, where the short rate appears explicitly as one of the modeled yield curve factors. Consequently, the short rate process can be parameterized directly and macroeconomic variables can be integrated according to a hypothesized central bank reaction function. For this reason, I denote the derived model as being short-rate based (SRB); (b) The model specification can be seen as an arbitrage-free version of Svensson and Söderlind (1997), but it relies only on one so-called 'shape-parameter' to establish the model-implied link between yield curve factors and observed yields, and the SRB model is therefore easy to estimate in comparison to the Svensson and Söderlind (1997) model. Christensen, Diebold, and Rudebusch (2009) show that an arbitrage-free five-factor generalization of their dynamic Nelson-Siegel framework provides a good fit to the original Svensson-Söderlind model (Svensson and Söderlind (1997)). Five factors are needed to match the loading structure, because the Svensson and Söderlind (1997) specification relies on two non-linear shape parameters: one that determines the shape of both the (negative) slope and the first curvature factor, and another that determines the shape of the second curvature factor. Such a loading structure can only be matched by an arbitrage-free model if two distinct eigenvalues are included in the dynamic evolution of the factors under the pricing measure. Instead of providing an exact match to the Svensson-Söderlind loadings, I show how to preserve the essential features of the model by using only one shape parameter (one eigenvalue under the pricing measure); (c) I demonstrate that the proposed model is a parameter-constrained member of JSZ family (JSZ2011) of GDTS models.

The SRB model is estimated on US data observed at a monthly frequency and covering the period from June 1961 to November 2017. It is shown to fit these data as well as the Svensson-Söderlind model does. A monetary policy rule is implemented such that the short rate process depends on the inflation rate and on industrial production. A similar parameterization cannot be obtained for the Svensson-Söderlind model, since this model does not include the short rate process explicitly among its yield factors. Instead, the macro series are included in a flexible way into the Svensson-Söderlind model to allow a policy rule to possibly emerge as a linear combination of the included variables. Impulse-response functions show that the SRB model produces an economically meaningful and statistically significant relationship between the short rate and the macroeconomic variables. A similar relationship cannot be recovered from the Svensson-Söderlind parametrization.

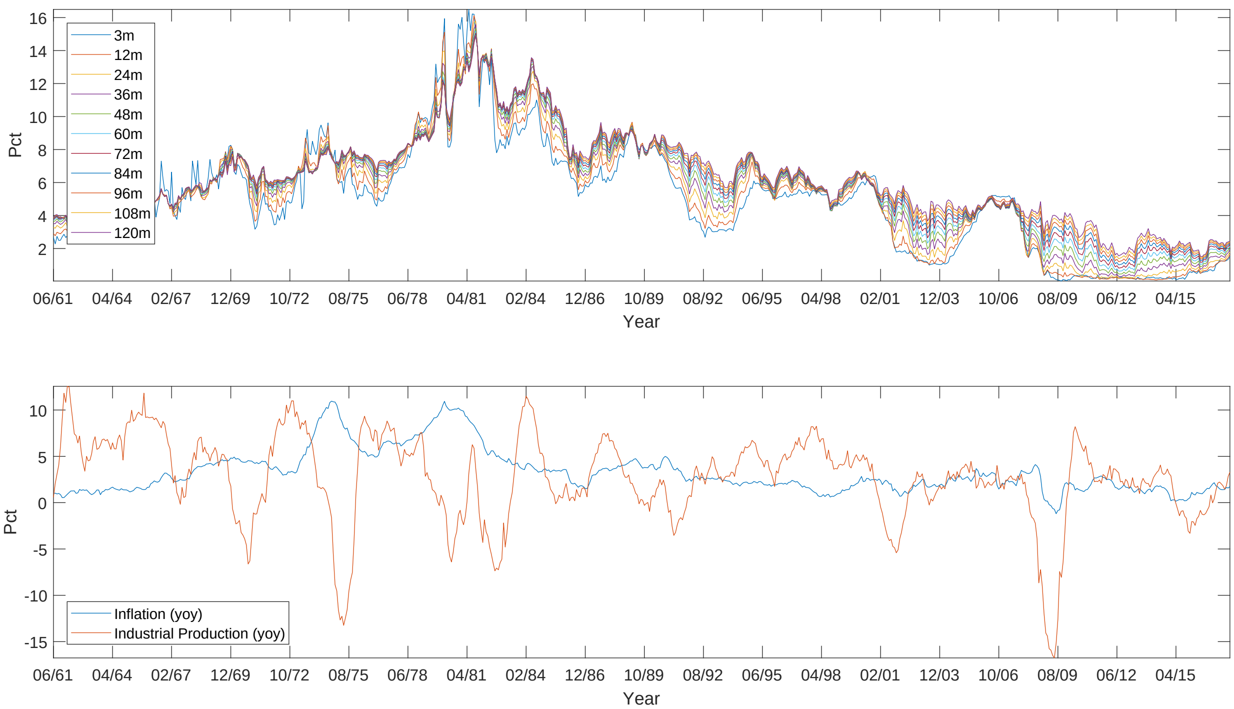
The rest of the paper is organized as follows. Section 2 gives an overview of the data used in the empirical part of the paper. Section 3 derives the model, Section 3.1 demonstrates how the SRB model is related to the JSZ model family, and Section 3.2 provides a yields-only comparison between the SRB and the Svensson and Söderlind (1997) models. Section 4 describes how the SRB model is estimated and extended to embed a monetary policy rule, where macroeconomic variables are included as unspanned factors (Joslin, Priebsch, and Singleton (2014), Bauer and Rudebusch (2014), and Coroneo, Giannone, and Modugno (2016), among others). Finally, Section 5 concludes the paper.

2 Data

The empirical work in this paper is done using the yield curve dataset prepared by Gurkaynak, Sack, and Wright (2006)[2] together with macroeconomic data downloaded from the

[2]Yield curve factors are downloaded from http://www.federalreserve.gov/pubs/feds/2006/200628/200628abs.html and converted into yields for the desired maturities using Svensson and Söderlind (1997) and Svensson (1994)

FRED database[3]. The data cover the period from June 1961 to November 2017 and are observed at a monthly frequency. Zero coupon yields at $\{3, 12, 24, \ldots, 120\}$ month maturities are used. The macroeconomic data series are industrial production and the PCE-index. Both series are converted into year-on-year percentage changes. Figure 1 gives a visual representation of the data used in the paper.



The upper panel shows the zero coupon yield curve data observed at maturities of $\{3, 12, 24, \ldots, 120\}$ months. The lower panel shows the year-on-year change in the PCE and industrial production indexes. Data are observed observed monthly, and cover the period from June 1961 to November 2017.

Figure 1: Yield curve and Macro Data

[3]Federal Reserve Bank of St. Louis, https://fred.stlouisfed.org/.

3 The short-rate based four-factor arbitrage-free term structure model

I rely on a standard affine set-up (see, e.g., Duffie and Kan (1996), Dai and Singleton (2000), and Ang and Piazzesi (2003)) to derive a discrete-time arbitrage-free model that has a loading structure similar to that of a dynamic Svensson and Söderlind (1997) model. Within the continuous-time setting Christensen, Diebold, and Rudebusch (2011) have shown how to maintain the parametric loading structure of the Nelson and Siegel (1987), while ensuring that arbitrage constraints are fulfilled.[4] Discrete-time versions of the same model have been derived previously (Niu and Zeng (2012) and Li, Niu, and Zeng (2012)). Christensen, Diebold, and Rudebusch (2011) show that five factors are needed to generate an arbitrage-free term structure model where the factor loadings match precisely those of Svensson and Söderlind (1997). Instead of providing an exact fit, I derive a parsimonious four-factor model with a closed-form loading structure that maintain the characteristics of the Svensson and Söderlind (1997) model while allowing for direct parametrization of the short rate process; in fact, the short rate appears as one modeled factors.

Let X_t denote the vector of the modeled yield curve factors, at time t. Furthermore, let the dynamics of X_t be governed by vector autoregressive (VAR) processes of order one, under both the empirical measure, $\mathbb{P}$, and the pricing measure, $\mathbb{Q}$:

$$X_t = \mu^{\mathbb{P}} + \Phi^{\mathbb{P}} \cdot \left(X_{t-1} - \mu^{\mathbb{P}}\right) + \Sigma\epsilon_t^{\mathbb{P}}, \qquad \epsilon_t^{\mathbb{P}} \sim N(0,1) \tag{1}$$

$$X_t = \mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}} \cdot \left(X_{t-1} - \mu^{\mathbb{Q}}\right) + \Sigma\epsilon_t^{\mathbb{Q}}, \qquad \epsilon_t^{\mathbb{Q}} \sim N(0,1). \tag{2}$$

with $\Sigma\Sigma' = \Omega$ being the variance of the residuals, which is the same for both measures. The risk free one-period short rate is assumed to be a function of X_t, such that:

$$i_t = \rho_0 + \rho_1' X_t. \tag{3}$$

To facilitate a mapping between the empirical and pricing measures, the time-varying

[4]See also, Krippner (2013) and Diebold and Rudebusch (2013).

market prices of risk is specified in the following way:

$$\lambda_t = \lambda_0 + \lambda_1 X_t, \tag{4}$$

with λ_0 being of dimension $N - by - 1$, and λ_1 being of dimension $N - by - N$, and where the mapping between $\mathbb{P}$ and $\mathbb{Q}$ is assumed to result from:

$$\lambda_0 \quad = \Sigma^{-1}\left(\mu^{\mathbb{P}} - \mu^{\mathbb{Q}}\right) \tag{5}$$

$$\lambda_1 \quad = \Sigma^{-1}\left(\Phi^{\mathbb{P}} - \Phi^{\mathbb{Q}}\right). \tag{6}$$

Finally, the yield at time t for maturity n is assumed to be an affine function of the factors, with a set of loading factors that depend on maturity:

$$y_{t,n} = -\frac{A_n}{n} - \frac{B_n'}{n} X_t. \tag{7}$$

The expressions for A_n and B_n can be found using the following set of recursive equations:[5]

$$A_{n+1} = A_n + B_n' \mu^{\mathbb{Q}} + \frac{1}{2} B_n' \Sigma \Sigma' B_n - \rho_0, \tag{8}$$

$$B_n = -\left[\sum_{k=0}^{n-1} \left(\Phi^{\mathbb{Q}}\right)^k\right]' \rho_1. \tag{9}$$

The above is well-know material from the existing term-structure literature.

In order to derive the four-factor SRB model I make two key assumptions. First, that the factors included in the model have the following economic interpretations:

$$X_t = \begin{bmatrix} \text{short rate} \\ \text{slope} \\ \text{curvature1} \\ \text{curvature2} \end{bmatrix}, \tag{10}$$

equation (3) then implies that $\rho_0 = 0$ and $\rho_1' = [1, 0, 0, 0]$. Second, it is assumed that the

[5]It is shown in e.g. Ang and Piazzesi (2003, Appendix A) and Mönch (2008, Appendix A) how the recursions can be derived.

$\mathbb{Q}$-measure dynamics are given by:

$$\Phi^{\mathbb{Q}} = \begin{bmatrix} 1 & 1-\gamma & 1-\gamma & 1-\gamma \\ 0 & \gamma & \gamma-1 & \gamma-1 \\ 0 & 0 & \gamma & \gamma-1 \\ 0 & 0 & 0 & \gamma \end{bmatrix}. \tag{11}$$

Closed-form expressions for the yield curve loadings can then be derived by first finding $\left(\Phi^{\mathbb{Q}}\right)^k$:

$$\left(\Phi^{\mathbb{Q}}\right)^k = \begin{bmatrix} 1 & 1-\gamma^k & -k\gamma^{k-1}(\gamma-1) & -\frac{k}{2}\gamma^{k-2}\left((k+1)\gamma^2-2k\gamma+k-1\right) \\ 0 & \gamma^k & k\gamma^{k-1}(\gamma-1) & \frac{k}{2}\gamma^{k-2}\left((k+1)\gamma^2-2k\gamma+k-1\right) \\ 0 & 0 & \gamma^k & k\gamma^{k-1}(\gamma-1) \\ 0 & 0 & 0 & \gamma^k \end{bmatrix}, \tag{12}$$

and then by substituting (12) into (9) obtains:

$$B_n = -\begin{bmatrix} n \\ \sum_{k=0}^{n-1} 1-\gamma^k \\ \sum_{k=0}^{n-1} -k\gamma^{k-1}(\gamma-1) \\ \sum_{k=0}^{n-1} -\frac{k}{2}\gamma^{k-2}\left((k+1)\gamma^2-2k\gamma+k-1\right) \end{bmatrix}. \tag{13}$$

Solving (13) gives:[6]:

$$B_n = -\begin{bmatrix} n \\ n-\frac{1-\gamma^n}{(1-\gamma)} \\ -n\gamma^{n-1}+\frac{1-\gamma^n}{(1-\gamma)} \\ -\frac{1}{2}n(n-1)(\gamma-1)\gamma^{n-2} \end{bmatrix}. \tag{14}$$

[6]The first entry of (13) follows immediately, the second entry uses $\sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x}$, the third and fourth entries can be found by consecutive substitution. For example, for n=5 the third entry of (13) is: $4\gamma^4-\gamma^3-\gamma^2-\gamma^1-\gamma^0$, which generalizes to $(n-1)\gamma^{n-1}-\sum_{k=0}^{n-2}\gamma^k$. Similarly, the fourth entry of (14) for n=5 is: $-(0+1(\gamma-1)\gamma^0+3(\gamma-1)\gamma^1+6(\gamma-1)\gamma^2+10(\gamma-1)\gamma^3)$, which generalizes to $-\frac{1}{2}n(n-1)(\gamma-1)\gamma^{n-2}$.

An expression for the yield curve at time t is then obtained if Y_t collects $y_{t,n}$ $\forall n$ by increasing maturity, and if $A = -A_n/n$ and $B = -B_n'/n$ are defined similarly. The expression for the yield curve observed at time t is then:

$$Y_t = A + BX_t + \eta_t. \tag{15}$$

Following, e.g. Gürkaynak and Wright (2012), the risk-free term structure can be calculated in the following way:

$$Y_{t,n_i}^{rf} = \frac{1}{n_i} \cdot E_t \left(\sum_{j=0}^{n_i-1} i_{t,t+j} \right). \tag{16}$$

where $Y_{n_i}^{rf}$ denotes the risk-free yield at a given maturity n_i. Hence, the risk-free yield is defined as the average of the accumulated short rate in (3), over a given investment horizon, for each observation point, t, covered by the sample. For the application in this paper, we use $n_i = 10$ years. Further, following Gürkaynak and Wright (2012), the term-premium is defined as the difference between the fitted yield curve and the risk-free curve, i.e.

$$TP_{t,n_i} = \hat{Y}_{t,n_i} - Y_{t,n_i}^{rf}. \tag{17}$$

Here, $Y_{t,n}$ refers to the model-fitted yield at time t for maturity n.

3.1 Relationship with the canonical GDTSM of Joslin, Singleton and Zhu (2011)

In the following I show that the SRB-model is a constrained member of the general family of Gaussian dynamic term structure model derived by Joslin, Singleton, and Zhu (2011) (JSZ).[7] Key to the JSZ model characterization is the idea of 'similar' matrices, known from linear algebra, where similarity is defined on the basis of the Jordan form. JSZ apply this idea to GDTSMs: if a given model's $\mathbb{Q}-$dynamics can be re-written in Jordan form, with

[7]This is not overly surprising since my model is a generalization of the arbitrage-free Nelson-Siegel model suggested by Christensen, Diebold, and Rudebusch (2011) (CDR), and since Joslin, Singleton, and Zhu (2011) show that the CDR model is a constrained member of the JSZ family.

distinct eigenvalues, then the model is identical (up to a rotation) to the JSZ canonical form.

Let J be the Jordan matrix, and V be a rotation matrix such that equation (11) can be reformulated as:

$$\Phi^{\mathbb{Q}} = V \cdot J \cdot V^{-1}. \tag{18}$$

Choosing V to be:

$$V = \begin{bmatrix} 1 & -(\gamma-1)^2 & \gamma-1 & -1 \\ 0 & \gamma^2 - 2*\gamma+1 & 1-\gamma & 1 \\ 0 & 0 & \gamma-1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{19}$$

implies that:

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & 1 & 0 \\ 0 & 0 & \gamma & 1 \\ 0 & 0 & 0 & \gamma \end{bmatrix}, \tag{20}$$

Since (20) is in Jordan form with repeated eigenvalues, there exists a mapping between the $\mathbb{Q}-$dynamics I propose above in (11) and the framework suggested by JSZ. The proposed SRB model is therefore a constrained member of the JSZ family of models.[8]

3.2 Relationship with the 4-factor Svensson-Söderlind model

Before embedding the monetary policy rule into the four-factor model, it is interesting to compare its characteristics and performance, on a yields-only basis, to that of Svensson and Söderlind (1997).

[8]The restriction of repeated eigenvalues, compared to the canonical JSZ form, is not rejected by the data used in the paper at a 5% level, using a likelihood ratio test.

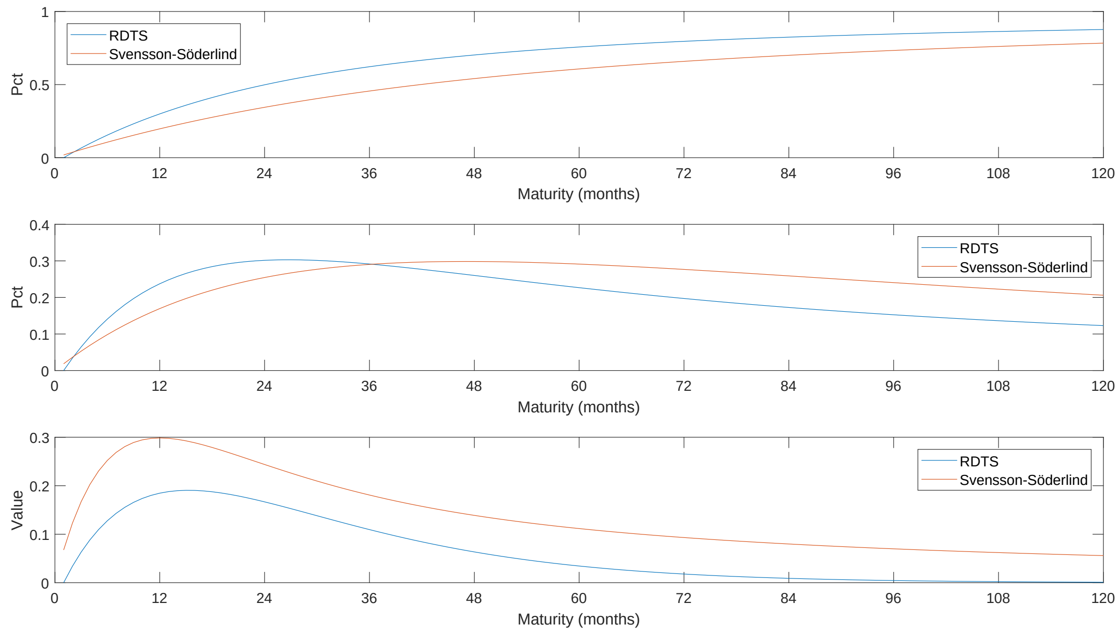
It is recalled that the Svensson-Soderlind loadings are given by:

$$H = \begin{bmatrix} 1 \\ \frac{1-e^{-\kappa_1 n}}{\kappa_1 n} \\ \frac{1-e^{-\kappa_1 n}}{\kappa_1 n} - e^{-\kappa_1 n} \\ \frac{1-e^{-\kappa_2 n}}{\kappa_2 n} - e^{-\kappa_2 n} \end{bmatrix}, \tag{21}$$

and that the loading structure of the SRB model is given by $B = B_n/n$, where B is given by equation (14):

$$B = -\begin{bmatrix} 1 \\ 1 - \frac{1-\gamma^n}{(1-\gamma)\cdot n} \\ -\gamma^{n-1} + \frac{1-\gamma^n}{(1-\gamma)\cdot n} \\ -\frac{1}{2}(n-1)(\gamma-1)\gamma^{n-2} \end{bmatrix}. \tag{22}$$

Figure 2 compares the loading structures of the Svensson and Söderlind (1997) and the SRB models. The shape parameter of the SRB model is $\gamma = 0.9324$, and the two Svensson-Söderlind shape parameters are estimated to be $\kappa_1 = 0.0381$ and $\kappa_2 = 0.1491$. These represent the optimal model estimates obtained from the data described in section 2. The loadings for the first factor are not shown as they equal 1 for both models across the included maturities. The first panel in the figure shows the loadings for the slope factor; and to facilitate easy comparison, the loading of this Svensson-Söderlind model is rotated to match that are the SRB model: let H_{slope} be the original slope loading for the Svensson-Söderlind model, panel 1 then plots $1 - H_{slope}$. The second and third panels compare the loadings for the first and second curvature loadings. The overall structure of the loadings from the two models is clearly similar. The first panel shows that the loadings for the slope factor have quite similar shapes, although the SRB loading assumes slightly higher values throughout the maturity spectrum, and also seems to arch upwards a bit more than the Svensson-Söderlind loading does. Level differences between the loading structures can naturally be subsumed by the corresponding factor values, so the shape attained by the loadings are of greater importance for the relative comparison between the models. Similarly, the second and the third panels show correspondence between the curvature loadings of the two models. Panel 2 indicates that the SRB model loading peaks around a maturity of 30 months, while the corresponding Svensson-Söderlind loading peaks around 40 month. This suggests that this Svensson-Söderlind factor allocates more weight to higher maturities compared to the SRB model, in terms of the first curvature loading.



The figure compares the loading structures of the Svensson and Söderlind (1997) and the SRB models. The shape parameter of the SRB model is $\gamma = 0.9324$, and the two Svensson-Söderlind shape parameters are estimated to be $\kappa_1 = 0.0381$ and $\kappa_2 = 0.1491$. The loadings for the first factor are not shown as they equal 1 for both models across the included maturities. The first panel shows the loadings for the slope factor, and to facilitate the comparison, the loading of this Svensson-Söderlind model is rotated to match that are the SRB model. Let H_{slope} be the original slope loading for the Svensson-Söderlind model, panel 1 then plots $1 - H_{slope}$. The second and third panels compare the loadings for the first and second curvature loadings.

Figure 2: Loading Structures

To test the impact of the minor differences in the loading structures documented above, the models are fitted to the US data outlined in section 2. Table 1 documents that both models produce very low root mean squared errors and that the added flexibility of the

Svensson-Söderlind model, via its reliance on two shape parameters, κ_1 and κ_2 (see equation (21)), as opposed to the one used by the SRB model (γ), gives it an economically insignificant edge of 1 basis points on average. The worst fitting maturity of the SRB model is the 12-month segment with a RMSE of 4.7 basis points, and the average RMSE across the eleven included maturities is 2.68 basis points. In comparison, the Svensson-Söderlind model produces the worst RMSE at the 24-months segment of 3.0 basis points, and the average RMSE is 1.58 basis points.

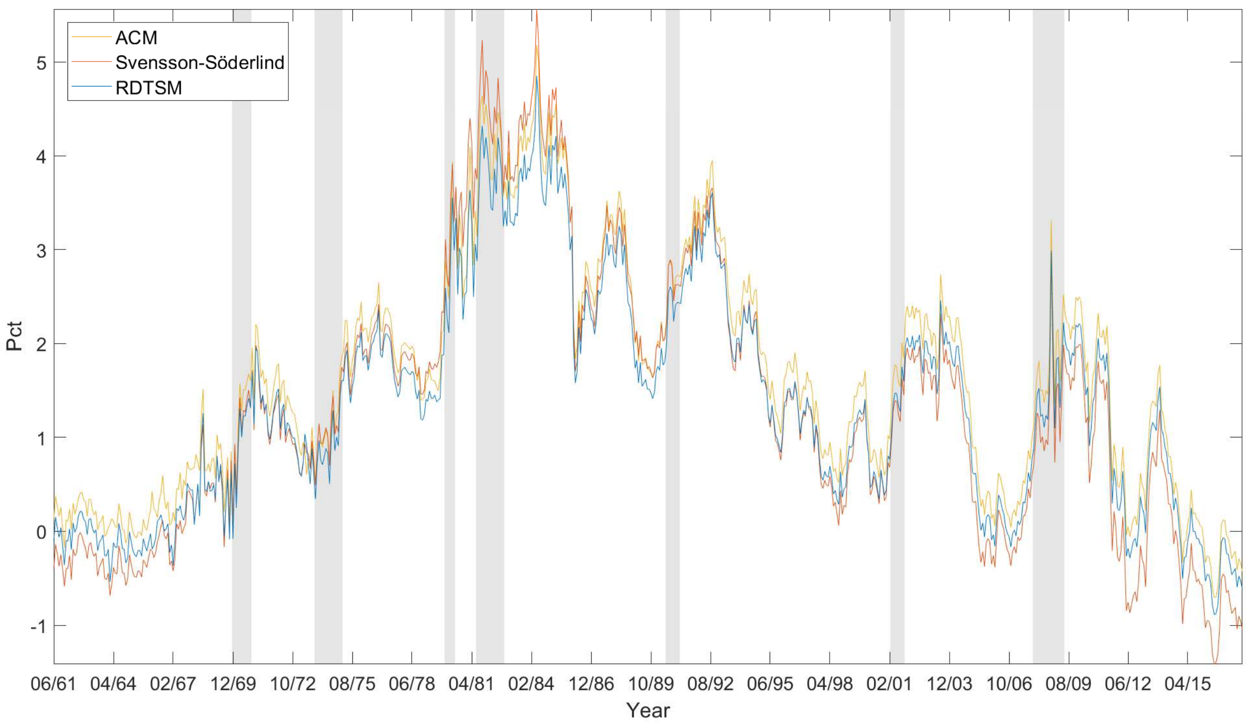
	3m	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
SRB Model	2.8	4.7	4.1	2.5	1.6	2.3	2.3	2.1	2.0	1.5	3.4
Svensson-Söderlind	0.3	2.6	3.0	2.0	0.7	1.4	1.9	1.6	0.8	0.7	2.3

The table shows the root mean squared errors in basis points of the SRB and the Svensson and Söderlind (1997) models when estimated using monthly US yield curve data covering the period from January 1961 to November 2017. Data are observed at maturities spanning 3 months to 10 years. The shape parameter of the SRB model is $\gamma = 0.9324$, and the two Svensson-Söderlind shape parameters are estimated to be $\kappa_1 = 0.0381$ and $\kappa_2 = 0.1491$.

Table 1: Root Mean Squared Errors (basis points)

In addition to in-sample fits, it is also interesting to gauge to what extent the two models produce similar policy relevant output. To this end Figure 3 reports the 10-year term premia obtained from yields-only versions (i.e. without including macroeconomic variables) of the two models. As a frame of reference, the figure also shows the 10-year term premia published by the Federal Reserve (New York) following Adrian, Crump, and Mönch (2013).[9] Visual inspection confirms that the dynamics and levels of the 10-year premium produced by the models are virtually identical.

[9] https://www.newyorkfed.org/research/data_indicators/term_premia



The figure compares 10-year term premia calculated by Adrian, Crump, and Mönch (2013), Svensson and Söderlind (1997) (with $\kappa_1 = 0.0381$ and $\kappa_2 = 0.1491$), and the SRB model (with $\gamma = 0.9687$) using US monthly US yield curve data covering the period from January 1961 to November 2016. Data are observed at maturities spanning 3 months to 10 years. Adrian, Crump, and Mönch (2013) term premia are downloaded from the Federal Reserve home page, and term premia from the other two model are calculated using equation (17). The shaded areas show NBER recessions.

Figure 3: Comparing Term Premia

4 Including a monetary policy reaction function

The discrete-time arbitrage-free model is estimated using a step-wise approach:[10]

1. Conditional on $\hat{\gamma}$, the risk-neutral factor dynamics, $\hat{\Phi}^{\mathbb{Q}}$, and the assumed affine structure, using equations (9), (11), and (13), a closed form expression can be found for the loading structure of the model, as shown in equation 22.

[10]The used calibration method can be seen as a special case of Andreasen and Christensen (2015) and Rios (2015).

2. With the factor interpretation in (15), I treat the factors as observed without error, and find them as $\hat{X} = \hat{B}^{-1}Y'$, where $\hat{B}^{-1}$ is the pseudo-inverse of $\hat{B}$.[11]

3. The dynamics under the $\mathbb{P}$-measure can now be found using (1).

4. With $\hat{X}$, $\hat{\Sigma}$, and $\hat{B}$ known, equations (8) and (15) are used to find $\hat{\mu}^{\mathbb{Q}}$ as the solution to $\min_{\mu^{\mathbb{Q}}} \sum_n \sum_t [Y_t - (\hat{A} + \hat{B}\hat{X}_t)]^2$.

5. Steps 1-4 are performed over a grid of values for γ, and the optimal $\hat{\gamma}$ is found as the one that minimises $\sum_n \sum_t \eta^2$.

Parameter constraints are imposed on the $\mathbb{P}$-dynamics of the models to implement the monetary policy rule. For the SRB model this is straight forward since the first yield curve factor is explicitly defined to be the short rate. Inflation and industrial production are therefore constrained to impact the first factor with one lag, and the autoregressive structure of the SRB model with a monetary policy rule therefore looks like this:

$$\Phi^{\mathbb{P}}_{\texttt{policy rule}} = \begin{bmatrix} \phi_{(1,1)} & 0 & 0 & 0 & \phi_{(1,5)} & \phi_{(1,6)} \\ \phi_{(2,1)} & \phi_{(2,2)} & \phi_{(2,3)} & \phi_{(2,4)} & 0 & 0 \\ \phi_{(3,1)} & \phi_{(3,2)} & \phi_{(3,3)} & \phi_{(3,4)} & 0 & 0 \\ \phi_{(4,1)} & \phi_{(4,2)} & \phi_{(4,3)} & \phi_{(4,4)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_{(5,5)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \phi_{(6,6)} \end{bmatrix}, \qquad (23)$$

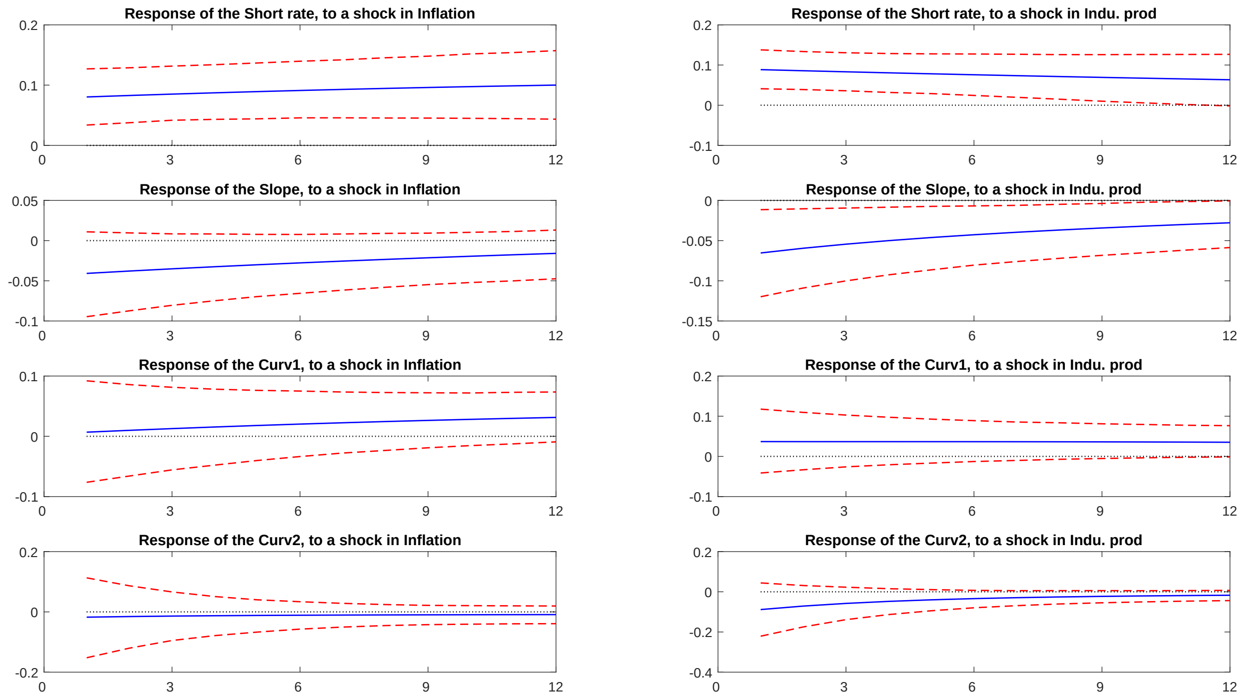
where the first row represents an empirical version of the policy rule, rows 2 to 4 allow for interaction between the yield curve factors, and rows 5 and 6 specify autoregressive processes for the macroeconomic variables. In comparison to Diebold, Rudebusch, and Aruoba (2006) I include the short rate process directly as one of the modelled yield curve factors. Typically in the term structure literature, as mentioned above, the level, slope, and curvature factors are used - or some variant hereof; even in the case when the factors are

[11]This is similar to Joslin, Singleton, and Zhu (2011) where it is assumed that a portfolio of yields form the underlying yield curve factors such that, in their notation, $\mathcal{P}_t \equiv \mathcal{W}y_t$, with $\mathcal{P}$ being the yield curve factors, $\mathcal{W}$ containing portfolio weights obtained via principal component analysis, and y are the observed yield curves.

treated as being latent, this is the economic interpretation that can be assigned to them on the basis of the estimated loading structure. Consequently, in macro economic applications of models using this factor identification strategy authors have resorted to including the short rate as an additional macro economic variable, see among others, Diebold, Rudebusch, and Aruoba (2006). The upshot of SRB model is that that the short rate factor enters directly as a priced factor, and that it can be parameterised directly (see equation (7). Therefore, the SRB model needs only to include two macroeconomic series as being unspanned (Joslin, Priebsch, and Singleton (2014) and Bauer and Rudebusch (2014)) and not the short rate process itself. In addition to this, the SRB avoids, by construction, any possible multicollinearity that may materialise between a linear combination of the traditional yield curve factors (level, slope and curvature), and the otherwise additionally included unspanned short rate process.

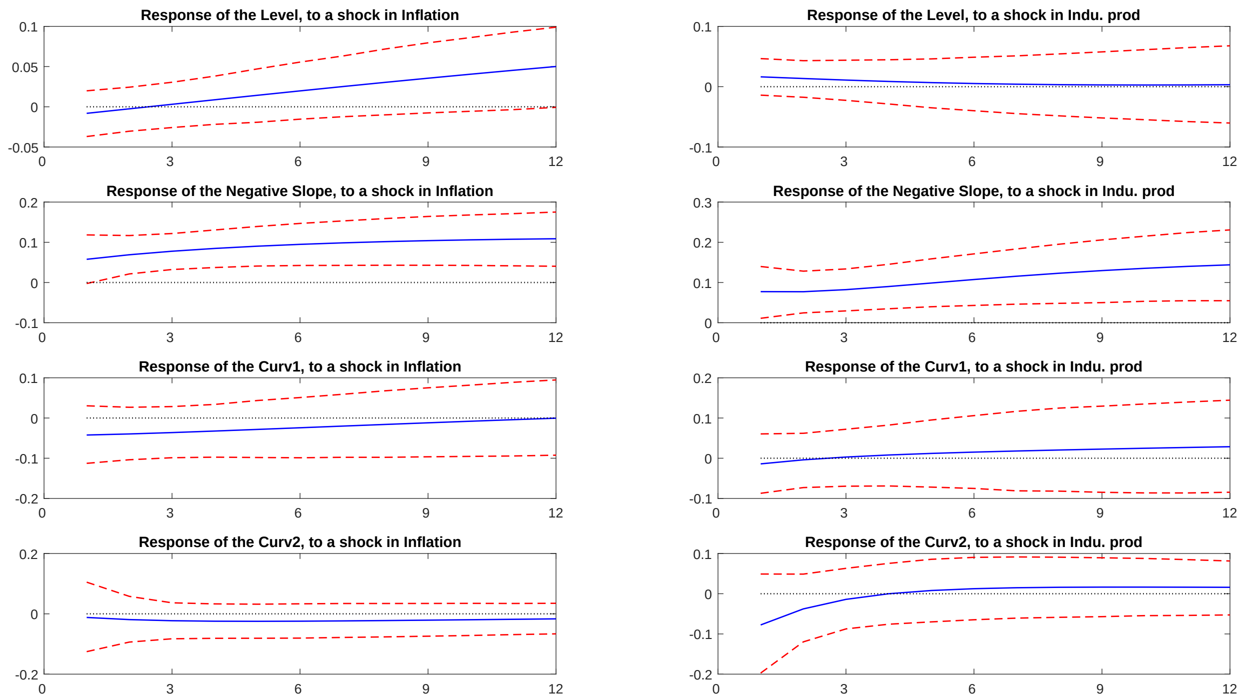
Implementing a monetary policy rule in the context of the Svensson-Söderlind model (Svensson and Söderlind (1997)) is not possible since the factor structure does not include a short rate factor directly. Instead, the model includes is built on a level, a (negative) slope, and two curvature factors. In principle, the short rate can be recovered, since the sum of the level and the (negative) slope equals the short rate. In empirical applications, a policy rule for the short rate may materialize, as a linear combination of the level and slope factors in conjunction with the included macroeconomic variables - however nothing will ensure this to happen at the outset. To test whether data lines up in a way where this occurs I estimate the Svensson-Söderlind model without imposing any constraints on the dynamic evolution of its yield curve factors.[12]

[12]Experimenting with various constraints did not produce results that were materially different from the ones shown in the text.



The figure shows generalized impulse-response functions (IRFs) following Pesaran and Shin (1998) over 1 to 12 months, for the Rotated Dynamic Term structure model outlined in Section 3. Each column contains the response of the yield curve factors to shocks in each of the macro economic variables. It is noted that this ordering does not affect the impulse-responses since generalized IRFs are used. The (blue) line shows the impulse-response and the dotted (red) lines shows the 95% confidence intervals obtained using $10,000$ parametric bootstrap samples. The macroeconomic variables are included into the model as unspanned factors.

Figure 4: Impulse-Response Functions: SRB model with unspanned macro variables



The figure shows generalized impulse-response functions (IRFs) following Pesaran and Shin (1998) over 1 to 12 months, for the Svensson and Söderlind (1997) model. It is noted that this ordering does not affect the impulse-responses since generalized IRFs are used. The (blue) line shows the impulse-response and the dotted (red) lines shows the 95% confidence intervals obtained using $10,000$ parametric bootstrap samples. The macroeconomic variables are included into the model as unspanned factors.

Figure 5: Impulse-Response Functions: Svensson-Söderlind model with unspanned macro variables

The Svensson-Söderlind model is estimated using OLS, with a grid-search algorithm determining the two shape parameters, κ_1 and κ_2.

Figure 4 and 5 show the generalized impulse-response functions (IRFs) (Pesaran and Shin (1998)) for the SRB and Svensson-Söderlind models, respectively. Only the responses of the yield curve factors to shocks in the two macroeconomic variables are shown, this is done to enhance the readability of the charts. Simulation based standard errors are reported in the figures (dotted red lines) together with the mean impact (blue line).

The impact of the policy rule is clear in the case of the SRB model. Figure 4 shows that the short rate reacts positively, as expected, to shocks in inflation and industrial production. Over a period of 12-months these impact are significantly different from zero, and relatively persistent. A positive reaction is also seen for the slope when industrial production is shocked. This impact is statistically significant and negative, however it is smaller in absolute magnitude than the corresponding impact seen in the short rate. Two opposite impacts are therefore captured by the SRB model. First, the short rate is positively related to inflation and productivity, as one would expect. However, the parallel shift upwards in the yield curve implied by the reaction of the short rate to shocks in industrial production is to some degree offset by a downward adjustment of the slope factor, in the form of a mild flattening of the curve. Since the magnitude of the short rate response is largest, there is an overall net upwards revision of the curve as a response to productivity shocks. The two curvature factors do not display statistically significant IRFs.

Figure 5 shows the impact in the context of the Svensson-Söderlind of shocking the two macroeconomic variables. Only a statistically significant impact is seen for the (negative) slope factor to shocks in inflation and industrial production. The positive reaction observed is consistent with the slope impacts seen in the SRB model, since the Svensson-Söderlind model is parameter with a negative slope.[13] The impact of either shock therefore leads to decrease in the over all level of the yield curve, since the observed flattening impetus from the slope factor, is not offset by an opposite and stronger movement in the level factor. In effect, a somewhat counter intuitive outcome is generated by the Svensson-Söderlind where positive shocks to inflation and industrial production lead to a lower and flatter yield curve.[14]

[13]The empirical counterpart of the slope factor in the SRB model is the long end of the yield curve minus the short end of the curve, e.g. a 10-year yield minus a 1-month yield. So, when the slope factor is positive in the SRB model, the yield curve is upwards sloping. The empirical counterpart of the slope in the Svensson-Söderlind model is the short end of the yield curve minus the long end of the curve, e.g. a 1-month yield minus a 10 year yields. So, when the slope factor in the Svensson-Söderlind model is positive, the yield curve is downward sloping.

[14]This conclusion also holds if the confidence levels on the IRFs are disregard, and one simply looks at the recorded impacts. The final impact on the yield curve of the produced impacts is calculated by multiplying the impacts by the respective factor loading shown in Figure 2.

5 Summary

I present a four factor discrete-time short-rate based (SRB) dynamic term structure model that empirically behaves like the Svensson-Söderlind model (Svensson and Söderlind (1997)) and where the loading structure is know in closed-form. In contrast to the popular Svensson-Söderlind model, the model set forth in the current paper is (1) arbitrage-free; (2) easy to estimate because it relies on only one non-linear shape parameter, where the Svensson-Söderlind model requires two such parameters; (3) includes a rotation of the yield curve factors such that the short rate appears directly as a priced yield curve factor, in addition to the slope and the two curvature factors; and (4) facilitates direct inclusion of a monetary policy rule, where macro economic aggregates impact the short rate process.

Using US yield curve data observed at a monthly frequency and covering the period from June 1961 to October 2016, the model is estimated and is shown to fit the data as well as the Svensson-Söderlind model. A monetary policy rule is implemented such that the short rate process in the SRB model depends on the inflation rate and industrial production. A similar parameterization cannot be obtained for the Svensson-Söderlind model, since this model does not include the short rate process explicitly among its yield factors. Instead, the macro series are included in a flexible way to allow the policy rule to emerge as a linear combination of the included variables. Impulse-response functions show that the SRB model produces an economically meaningful and statistically significant relationship between the short rate and the macroeconomic variables. A similar relationship cannot be recovered from the Svensson-Söderlind parametrization.

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